Linear algebra - Solutions to practice problems

1. Compute the following, or state that it is undefined.

(a) $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$

(b) Not defined.

(c) $\begin{bmatrix} 17 & 8 \\ 19 & -8 \end{bmatrix}$

(d) Not defined.

(e) Not defined.

(f) The inverse is $\begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & -7 \end{bmatrix}$.

(a) A basis is $\left\{ \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} \right\}$. (any non-zero multiple of this vector is also a basis)

(b) The solutions are

$$\mathbf{x} = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix} + r \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix},$$

where r is any real number.

(c) The rank of A is 2.

(d) The nullspace is the span $\text{sp}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} \right)$. (same as part (a))

(e) A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix} \right\}$.

3 Let $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$ and suppose that the reduced row-echelon form of A is $H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (you can check this if you want to practice row reduction).

(a) A basis for the row space is $\left\{ \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$.

(b) A basis for the column space is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

(c) A basis for the nullspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(d) The rank of A is 3.

(e) Not every system $A\mathbf{x} = \mathbf{b}$ is consistent. When we row reduce $[A \,|\, \mathbf{b}] \sim [H \,|\, \mathbf{c}]$, the last entry of $\mathbf{c}$ has to vanish. In terms of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ this means that the system is only consistent if $-b_2 - b_3 + b_4 = 0$ (to see this you have to row reduce $[A \,|\, \mathbf{b}]$).

4. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$.

(a) $T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

(b) The standard matrix representation of T is $A = \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ -5 & 3 \end{bmatrix}$.

(c) The rank of T is 2.

(d) The kernel of T is $\{\mathbf{0}\}$. (we know that $\operatorname{rank}(T) + \dim\ker(T) = 2$, so part (c) tells us that the kernel is zero dimensional)

5. Write down the standard matrix representations for the following linear transformations.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

6. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors in $\mathbf{R}^5$.

(a) A basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Let $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_1 - \mathbf{v}_3$.

(i) No, $\mathbf{w}_1 - \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$, so $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are not linearly independent.

(ii) Yes. Since $\mathbf{w}_3 = \mathbf{w}_1 - \mathbf{w}_2$, we have $\text{sp}(\mathbf{w}_1, \mathbf{w}_2) = \mathbf{sp}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Also, $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent since if

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 = \mathbf{0},$$

then

$$r_1(\mathbf{v}_1 + \mathbf{v}_2) + r_2(\mathbf{v_2} + \mathbf{v_3}) = \mathbf{0},$$

so

$$r_1\mathbf{v}_1 + (r_1 + r_2)\mathbf{v}_2 + r_2\mathbf{v}_3 = \mathbf{0}.$$

But $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, so we must have $r_1, r_2 = 0$.

(iii) $2\mathbf{v}_1 + \mathbf{v_2} - \mathbf{v}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$.

7. Which of the following are subspaces of $\mathbf{R}^3$? For those which are subspaces, find a basis.

(a) Not a subspace since $[1, 0, 1], [0, 1, 1]$ are in the set, but their sum $[1, 1, 2]$ is not in the set.

(b) Not a subspace since $[1, -1, 0]$ is in the set, but the scalar multiple $[-1, 1, 0]$ is not in the set. .

(c) This is a subspace with basis $\{[1, 0, 0], [0, 1, 0]\}$. .

8.

(a) Yes. To see this, row reduce the matrix whose columns are the given vectors.

(b) It is not possible. If you form the matrix A with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and row reduce, there will be a column with no pivot since A has only 3 rows (the vectors are in $\mathbf{R}^3$).

(c) For example $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [1, 1, 1, 0]$. (there are many other possibilities)